Analize III, pismeni ispit, 01.09.2014.

1. Ispitati neprekidnost funkcije $f(x,y) = \begin{cases} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2}, & (x,y) \neq (1,0) \\ 0, & (x,y) = (1,0) \end{cases}$.

2. Oblast D je ograničena pravama $2x-y=1$, $2x-y=3$, $x+y=-2$ i $x+y=0$. Dati integral $\iint\limits_D xy\,\mathrm{d}x\mathrm{d}y$ izračunati na dva načina:

(a) bez uvođenja smjena promjenjivih; (b) uvođenjem smjena promjenjivih.

3. Izračunati pomoću Stoksove formule (ili direktno) $I = \oint\limits_c y^2\mathrm{d}x + z^2\mathrm{d}y + x^2\mathrm{d}z$, ako je c kontura trougla $\triangle ABC$, $A(2;0;0)$, $B(0;1;0)$, $C(0;0;-3)$, pređena u pozitivnom smislu.

4. (30%)(a) Primjenom formule Gauss-Ostrogradskog izračunati površinski integral druge vrste $\oiint\limits_W (z+1)\mathrm{d}x\mathrm{d}y$ gdje je W-sfera $x^2+y^2+z^2=R^2$.

(70%)(a) Neka je data funkcija $f:\mathbb{R}^3 \to \mathbb{R}$. Ako je $\vec{r}$ radijus vektor i $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplaceov operator, dokazati da vrijede sljedeće dvije jednkosti

$$\Delta(x \cdot f) = 2\frac{\partial f}{\partial x} + x \cdot \Delta(f), \qquad \mathrm{grad}(f) = \frac{1}{2}(\Delta(\vec{r} \cdot f) - \vec{r} \cdot \Delta(f)).$$

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Prije rješenja prepisati postavku (tekst) zadatka. Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

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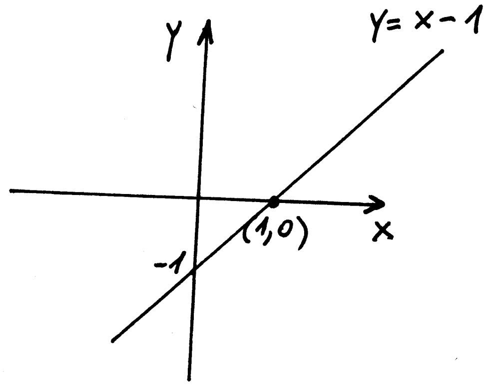
R: Jedina sumnjiva tačka u kojoj f-ja može imati prekid je tačka $(1,0)$. F-ja će biti neprekidna u ovoj tački akko

$$\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0)$$



tj. akko $\lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$.

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=0$.

$$\lim_{(x,0)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+0^2} = \lim_{(x,0)\to(1,0)} \ln x = 0$$

Posmatrajmo približavanje tački $(1,0)$ preko prave $y=x-1$

$$\lim_{(x,x-1)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+(x-1)^2} = \lim_{(x,x-1)\to(1,0)} \frac{\ln x}{2} = 0$$

Odavde možemo naslutiti da je možda vrijednost ovog limesa u tački $(1,0)$ jednaka 0.

Prisjetimo se teoreme "dva policajca":

$$\forall (x,y)\in\mathbb{R}^2 \quad g(x,y)\leq f(x,y)\leq h(x,y) \text{ i } \lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = M$$

$$\Rightarrow \lim_{(x,y)\to(a,b)} f(x,y) = M \iff |f(x,y)-M|\to 0,\ (x,y)\to(a,b)$$

$(x+1)^2 \geq 0 \quad \forall x\in\mathbb{R}$

$(x-1)^2+y^2 \geq 0 \quad \forall x,y\in\mathbb{R}$

$$(x-1)^2+y^2 \geq (x+1)^2 \Rightarrow 0 \leq \frac{(x-1)^2}{(x-1)^2+y^2} \leq 1 \Rightarrow$$

$$\Rightarrow 0 \leq \frac{(x-1)|\ln x|}{(x-1)^2+y^2} \leq |\ln x| \to 0,\ (x,y)\to(1,0) \xRightarrow[\text{dva polic.}]{\text{Teor.}} \lim_{(x,y)\to(1,0)} \frac{(x-1)^2 \ln x}{(x-1)^2+y^2} = 0$$

F-ja je neprekidna.

(#) Oblast D je ograničena pravama $2x-y=1$, $2x-y=3$, $x+y=-2$ i $x+y=0$. Dati integral

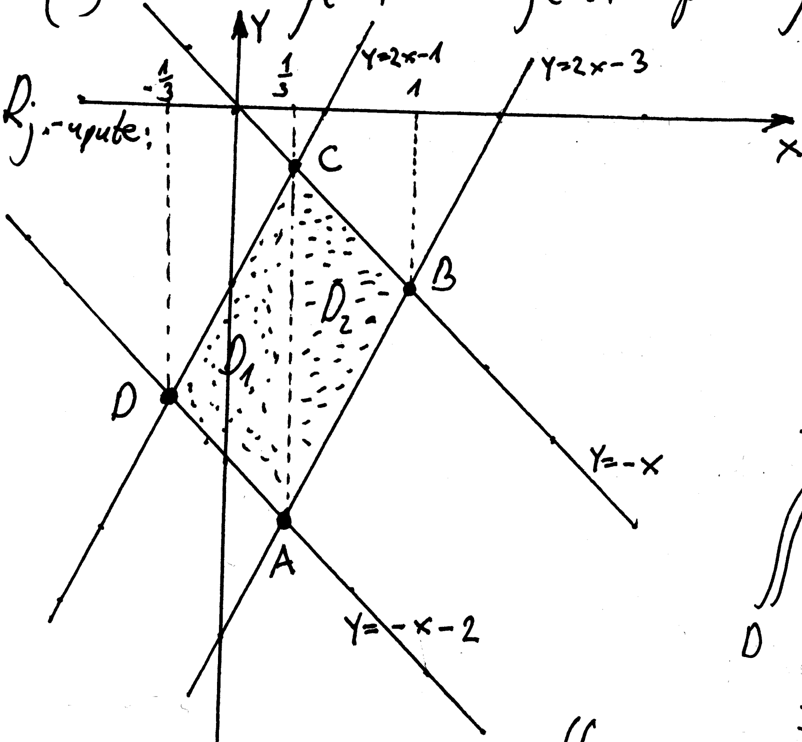
$$\iint_D xy\,dx\,dy$$

izračunati na dva načina

(a) bez uvođenja smjena promjenjivih;

(b) uvođenjem smjena promjenjivih.

Rj.-upute:



Presječne tačke pravih su

$A\left(\frac{1}{3}; -\frac{7}{3}\right)$, $B(1;-1)$,

$C\left(\frac{1}{3}; -\frac{1}{3}\right)$, $D\left(-\frac{1}{3}; -\frac{5}{3}\right)$

I način: (a)

$$\iint_D xy\,dx\,dy = \iint_{D_1\cup D_2} xy\,dx\,dy = \iint_{D_1} xy\,dx\,dy + \iint_{D_2} xy\,dx\,dy$$

$$\iint_{D_1} xy\,dx\,dy = \int_{-\frac{1}{3}}^{\frac{1}{3}} x\,dx \int_{-x-2}^{2x-1} y\,dy = \ldots = -\frac{8}{81} \quad \ldots(1)$$

$$\iint_{D_2} xy\,dx\,dy = \int_{\frac{1}{3}}^{1} x\,dx \int_{2x-3}^{-x} y\,dy = \ldots = -\frac{4}{9} \quad \ldots(2)$$

(1) i $(2) \Rightarrow \iint_D xy\,dx\,dy = \frac{-44}{81}$

II način (b)

uvodimo smjene promjenjivih $u = u(x,y) = 2x - y$

$v = v(x,y) = x + y$

Tada $D \xrightarrow{\text{transform.}} D'$: $\begin{cases} 1 \le u \le 3 \\ -2 \le v \le 0 \\ du\,dv = |J|\,dx\,dy \end{cases}$

$$J = \frac{D(x,y)}{D(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

Znamo da vrijedi : $J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = 3$.

$\Rightarrow \quad J = \frac{1}{3}$

$$\begin{aligned} u &= 2x - y \\ v &= x + y \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{1}{3}(u+v) \\ y &= \frac{1}{3}(-u+2v) \end{aligned}$$

Sad nije teško naći vrijednost datog integrala

$$I = \iint_D xy\,dx\,dy = \iint_{D'} \frac{1}{9}(u+v)(2v-u) \cdot \frac{1}{3}\,du\,dv =$$

$$= \frac{1}{27} \int_1^3 \left[\int_{-2}^{0} (u+v)(2v-u)\,dv \right] du = -\frac{44}{81}$$

Izračunati pomoću Stoksove formule (ili direktno)

$$I = \oint\limits_C y^2 dx + z^2 dy + x^2 dz,$$ ako je c kontura trougla ΔABC,

$A(2,0,0)$, $B(0,1,0)$, $C(0,0,-3)$, pređena u pozitivnom smislu.

Rj. <u>I način</u>

Prisjetimo se

$$\int\limits_C P_{(x,y,z)}dx + Q_{(x,y,z)}dy + R_{(x,y,z)}dz = \iint\limits_S \begin{vmatrix} dydz & dxdz & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

površinski integral druge vrste

$$\frac{\partial Q}{\partial x} = 0, \quad \frac{\partial P}{\partial y} = 2y \qquad \frac{\partial R}{\partial x} = 2x, \quad \frac{\partial P}{\partial z} = 0$$

$$\frac{\partial R}{\partial y} = 0, \quad \frac{\partial Q}{\partial z} = 2z$$

$$I = \oint\limits_C y^2 dx + z^2 dy + x^2 dz = \begin{vmatrix} \text{Formula} \\ \text{Stoksa} \end{vmatrix} = \iint\limits_S -2z\,dydz - 2x\,dxdz - 2y\,dxdy$$

gdje S predstavlja dio površi u prostoru ograničen trouglom ΔABC.

$A(2,0,0)$ Jednačina ravni kroz tri tačke $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$

$B(0,1,0)$

$C(0,0,-3)$

$$\begin{vmatrix} x-2 & y & z \\ -2 & 1 & 0 \\ -2 & 0 & -3 \end{vmatrix} = 0$$

koji je najlakši način da zapamtimo ovu formulu?

$$-3(x-2) - 6y + 2z = 0$$

$$-3x - 6y + 2z + 6 = 0$$

$\vec{n} = (-3, -6, 2)$

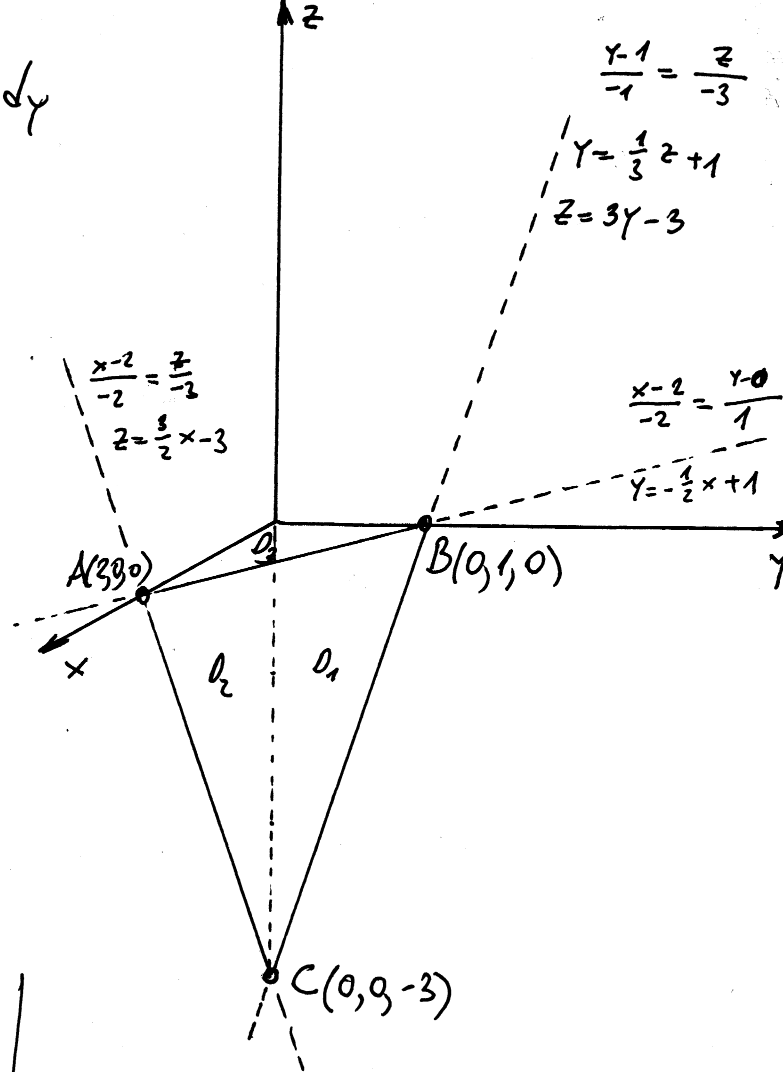
$|\vec{n}| = \sqrt{9+36+4} = 7$ $\qquad \vec{n}_0 = \left(-\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}\right)$

$$I=(-2)\iint\limits_S z\,dy\,dz + x\,dx\,dz + y\,dx\,dy$$

$$I_1=\iint\limits_S z\,dy\,dz = \left|\begin{array}{l}\cos\alpha<0\\ D_1:\begin{cases}0\le y\le 1\\ 3y-3\le z\le 0\end{cases}\end{array}\right| =$$

$$=-\iint\limits_D z\,dy\,dz = -\int_0^1 dy\int_{3y-3}^{0} z\,dz =$$

$$= \dots = +\frac{3}{2}$$



$$I_2=\iint\limits_S x\,dx\,dz = \left|\begin{array}{l}\cos\beta<0\\ D_2:\begin{cases}0\le x\le 2\\ \frac{3}{2}x-3\le z\le 0\end{cases}\end{array}\right|$$

$$= -\int_0^2 x\,dx\int_{\frac{3}{2}x-3}^{0} dz = \dots = -2$$

$$I_3=\iint\limits_S y\,dx\,dy = \left|\begin{array}{l}\cos\gamma>0\\ D_3:\begin{cases}0\le x\le 2\\ 0\le y\le -\frac{1}{2}x+1\end{cases}\end{array}\right| = +\int_0^2 dx\int_0^{-\frac{1}{2}x+1} y\,dy = \dots = \frac{1}{3}$$

Prema tome $I=(-2)\left(\frac{3}{2}-2+\frac{1}{3}\right)=\frac{1}{3}$ tražena vrijednost

II način

Koristimo formulu

$$\int\limits_C P(x,y,z)\,dx+Q(x,y,z)\,dy+R(x,y,z)\,dz=\iint\limits_S \begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix} dS$$

$$I=\oint_C y^2\,dx+z^2\,dy+x^2\,dz=\left|\begin{matrix}\text{formula}\\ \text{Stoksa}\end{matrix}\right|=$$

$$=\iint_S(-2z\cos\alpha-2x\cos\beta-2y\cos\gamma)\,dS=\left|\begin{matrix}\cos\alpha=-\frac{3}{7}\\ \cos\beta=-\frac{6}{7}\\ \cos\gamma=\frac{2}{7}\end{matrix}\right|$$

$$=\iint_S\left(\frac{6}{7}z+\frac{12}{7}x-\frac{4}{7}y\right)dS=\left|\begin{matrix}\text{ravan kroz tačke } A,B,C\\ -3x-6y+2z+6=0\\ 2z=3x+6y-6\\ z=\frac{3}{2}x+3y-3\end{matrix}\right.$$

označimo sa D projekciju oblasti S na xOy ravan

$$dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

$$\frac{\partial z}{\partial x}=\frac{3}{2}\qquad dS=\sqrt{\frac{49}{4}}=\frac{7}{2}$$

$$\frac{\partial z}{\partial y}=3$$

$$=\frac{7}{2}\iint_D\left(\frac{6}{7}\left(\frac{3}{2}x+3y-3\right)+\frac{12}{7}x-\frac{4}{7}y\right)dx\,dy$$

$$=\frac{7}{2}\iint_D\left(3x+2y-\frac{18}{7}\right)dx\,dy=\frac{7}{2}\int_0^2 dx\int_0^{-\frac{1}{2}x+1}\left(3x+2y-\frac{18}{7}\right)dy=\frac{7}{2}\cdot\frac{2}{21}=\frac{1}{3}$$

tražena vrijednost

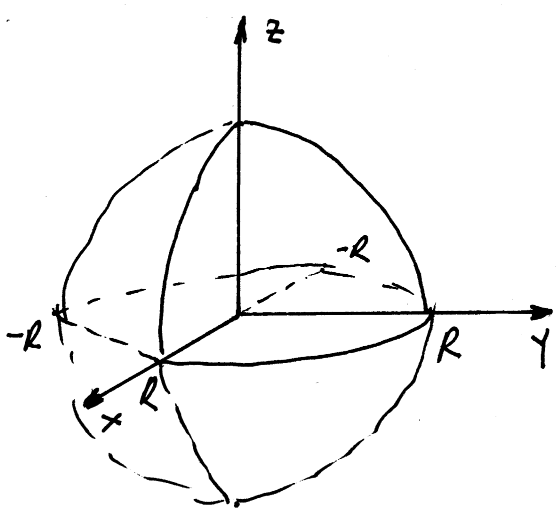
(#) Primjenom formule Gauss-Ostrogradskog izračunati površinski integral druge vrste $\oiint\limits_{W}(z+1)\,dx\,dy$ gdje je W - sfera $x^2+y^2+z^2=R^2$.

Rj. Znamo da $\underline{\oiint\limits_{S} P\,dy\,dz+Q\,dx\,dz+R\,dx\,dy} = \underline{\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\,dy\,dz}$

formula Gauss-Ostrogradskog

U našem slučaju:

$R(x,y,z)=z+1 \Rightarrow \frac{\partial R}{\partial z}=1$



$\oiint\limits_{W}(z+1)\,dx\,dy = \left|\begin{array}{l}\text{formula}\\ \text{Gauss-Ostrogradskog}\end{array}\right|$

$= \underbrace{\iiint\limits_{\Omega} dx\,dy\,dz}_{\text{zapremina kugle poluprečnika } R} = \frac{4}{3}R^3\pi$ traženo rješenje

(#) Neka je data f-ja $f: \mathbb{R}^3 \to \mathbb{R}$.
Ako je $\vec{r}$ radijus vektor i $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ Laplaceov operator dokazati da vrijede sljedeće dvije jednakosti

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Rj.
Obe jednakosti se mogu direktno dokazati izračunavanjem obe strane navedenih jednakosti.

$$\Delta(x \cdot f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(x \cdot f) =$$

$$= \underbrace{\frac{\partial}{\partial x}\left(f + x\frac{\partial f}{\partial x}\right)}_{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x}} + x\frac{\partial^2 f}{\partial y^2} + x\frac{\partial^2 f}{\partial z^2} =$$

$$= 2\frac{\partial f}{\partial x} + x \cdot \Delta(f).$$

Prvu jednakost označimo sa (1).

Time smo dokazali prvu jednakost $\Delta(x \cdot f) = 2\frac{\partial f}{\partial x} + x\,\Delta(f)$ ...(1)

$$\Delta(\vec{r} \cdot f) = \Delta\left((x,y,z) \cdot f\right) = \left(\Delta(xf), \Delta(yf), \Delta(zf)\right) \overset{(1)}{=}$$

$$\overset{(1)}{=} \left(2\frac{\partial f}{\partial x} + x \cdot \Delta(f), 2\frac{\partial f}{\partial y} + y \cdot \Delta(f), 2\frac{\partial f}{\partial z} + z \cdot \Delta(f)\right)$$

$$= 2\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) + (x,y,z) \cdot \Delta(f) = 2\,\text{grad}(f) + \vec{r} \cdot \Delta(f)$$

Odavde slijedi druga tražena jednakost

$$\text{grad}(f) = \frac{1}{2}\left(\Delta(\vec{r} \cdot f) - \vec{r} \cdot \Delta(f)\right)$$